Design of Nonlinear Stochastic Discrete-time System Controller

Guosheng Tang*, Mingxin Zhu, Wei Chen

School of Mathematics, Jiangsu University of Science and Technology, Zhangjiagang 215600, China

Abstract

Targeted at the optimal control of nonlinear stochastic discrete-time systems disturbed by additive noise under the quadratic performance index, this paper studies the design method of nonlinear stochastic discrete-time system controller, designs a controller to regulate nonlinear stochastic systems, and verifies the effectiveness of the controller through simulation.

Keywords: nonlinear system; optimal control; fuzzy control framework; dual control

1. INTRODUCTION

The linear–quadratic-Gaussian (LQG) control has raised an increasing concern among scholars of control theory and engineering as it can be widely applied in engineering and easily resolved mathematically. Targeted at linear stochastic systems, the classical LQG control problem seeks the mean of the sum of state and control in the quadratic forms. The LQG controller is also fundamental to the optimal control of perturbed non-linear systems. Probing into a class of nonlinear stochastic discrete-time systems, literatures (Zhang et al., 2008 Li et al., 2002; Carew et al., 1973; Jian et al., 2008) put forward an adaptive dual control method, aiming to obtain abundant estimation information, reduce some uncertainties in the systems, and thereby achieve better control performance. For a category of polynomial type nonlinear stochastic complex systems with polynomial shape, literature (Ian, 2007) adopts the iterative linearization strategy to get the suboptimal control of the original complex system under the framework of LQG, and proves the convergence of the algorithm. Fuzzy control system is considered as one of the most effective ways to examine complex nonlinear systems. The basic idea is to simplify the complex system with fuzzy rules, i.e., to approximate the original complex system with a series of linear systems, the number of which equals number of fuzzy rules. In the fuzzy rules, the antecedent is a variable easy to be measured in engineering, and the consequent is a simple linear system. Owing to its universality, the complex system processing approach has been widely concerned by engineers and theorists. So far, it has managed to solve a host of engineering and theoretical problems, giving birth to countless new methods (Piotr and Leslaw, 2010; Dong and Yang, 2011; Tuan et al., 2001). With the purpose of stabilizing close-loop systems, these novel methods bear some common characteristics. For instance, the controllers are extremely conservative for it is a must to find a common Lyapunov matrix for all local linear models. In order to overcome the shortcoming, numerous countermeasures come into being, such as the fuzzy Lyapunov function method (Liu and Zhang, 2003; Tanaka et al., 2003; Wang et al., 2007), the segmented Lyapunov function method (Cuerra and Vermeriren, 2004; Johansson et al., 1997) and methods based on the shape and structure of the membership function.

Literature (Sala and Arino, 2008) digs into a class of complex systems under the maximum-minimum framework and implements the optimal control in the worst case. However, the results are valid only if the system state can be obtained in real time. Literature (Chen et al., 2009) follows global linearization and T-S fuzzy rules to design a scholastic system filter under the LMI framework, failing to take account of the control problem. Some meaningful results have been obtained through the research on the $H_2/H_{\infty}$ control problem of linear systems with state-dependent noise. The T-S model is, in essence, the design of fuzzy rules and the selection of consequent system model. Different models must differ in closed-loop performance. For the convenience of mathematical processing, priorities should be given to the simplest model. In the extreme situation, the simplest model is the linear model. Nevertheless, the simplification is conducted at the sacrifice of the performance of closed-loop system.
System complexity is identical twin of uncertainty. If the statistical characteristics of the system uncertainty is known prior to the test, the LQG control will be the best option for the consequent model of fuzzy rules. In this paper, the Takagi-Sugeno fuzzy control model is established to solve the control problem of complex systems disturbed by additive white Gaussian noise interference. The model uses the Kalman filter to estimate the system’s state information, and obtains the control gain by dynamic programming. Designed specifically for nonlinear stochastic complex systems, the resulting controller is learning-capable and less conservative, which ensures that the closed-loop system has the desired performance indices, and enables the control and filtering functions in the control loop. In contrast, literature (Sala and Arino, 2008) only deals with system control and does not enable the filtering function in the control loop; literature (Chen et al., 2009) only solves the filtering problem of stochastic systems.

2. PROBLEM DESCRIPTION

The following discrete-time dynamical system is taken into consideration:

\[ x(k+1) = f(x(k), u(k)) + w(k) \]  \hspace{1cm} (1)

\[ y(k) = h(x(k)) + v(k) \]  \hspace{1cm} (2)

Where \( x(k) \in \mathbb{R}^n \) is the state vector; \( u(k) \in \mathbb{R}^m \) is the input vector; \( y(k) \in \mathbb{R}^r \) is the measurement vector; \( w(k) \in \mathbb{R}^n \) and \( v(k) \in \mathbb{R}^r \) are uncorrelated process noise and measurement noise, respectively; \( f \) and \( h \) are the nonlinear functions in the \( n \)-th and \( r \)-th dimensions, respectively; \( f \) and \( h \) are either unknown or known.

The performance will differ inevitably if the systems are controlled by different input sequences \( \{u(k)\} \). Thus, the performance indices of the control sequence is determined by the functional evaluation as below. In other words, the performance indices are:

\[ J = x^T(N)Q_0x(N) + \sum_{k=0}^{N-1} [x^T(k)Q_1x(k) + u^T(k)Q_2u(k)] \]  \hspace{1cm} (3)

Where \( k=0 \) and \( N \) stand for the initial time and the terminal time of the control process, respectively; \( Q_0 \) and \( Q_1 \) are semi-positive definite matrices, and \( Q_2 \) is a positive definite matrix; the three matrices, time-invariant or time-variant, have appropriate dimensions and share exactly the same processing method. For the sake of simplicity, this paper only discusses the time-invariant situation.

The performance index formula (3) weighs the cumulative error of the target against the cumulative energy required for the control in the whole control process. The semi-positive definiteness of \( Q_0 \) and \( Q_1 \), together with the positive definiteness of \( Q_2 \), guarantees that the control index function \( J \) is convex, thereby ensuring the global optimality of the control.

In addition to the known prior information, e.g. the statistical characteristics of \( w(k) \) and \( v(k) \), the controller applies additive interferences to the nonlinear stochastic systems (1) and (2) at the \( k \)-th moment. Before that, the control input sequence \( \{u(0), u(1), \ldots, u(k-1)\} \) has been applied to the systems until the output \( \{y(1), y(2), \ldots, y(k)\} \) is observed at the current moment. Therefore, the real-time information set \( I^k \) available to the controller at the \( k \)-th moment is defined as:

\[ I^k = \{u(0), \ldots, u(k-1); y(1), \ldots, y(k)\} \]

The current task is to find a control law in the form of \( u(k) = \mu_k(I^k) \) to minimize the value of the performance index (3) in the statistical sense, that is,
\[
(P) \quad \min_{[u(k)]} E\{J\} \\
\text{s.t.} \quad x(k+1) = f(x(k),u(k)) + w(k), k = 0,1,2,\cdots,N-1 \\
y(k) = h(x(k)) + v(k), k = 1,2,\cdots,N 
\]

Where \(E\{\cdot\}\) is the expectation operator.

At the current moment, \(\hat{F}\) is the only system information known to the controller. This requires that the control law must be constrained by the form of \(u(k) = \mu_k(\hat{F})\). The form is physically achievable and allowed. The said constraint of the control law further complicates the solution of problem \((P)\), making it impossible to obtain the analytical solution of most systems. However, if the object is a linear stochastic system and the process noise, measurement noise, the initial state are uncorrelated white Gaussian noises, the control of problem \((P)\) is essentially the classic LQG control, and the online computational load can be greatly eased following the separation principle. Inspired by the above analysis, the T-S fuzzy method is adopted to segment and linearize the complex system, and to approximate the control problem \((P)\) with several local LQG problems. The strategy both simplifies problem \((P)\), making it resolvable under the LQG framework, and retains the known prior information like the statistical characteristics of stochastic interference.

For the nonlinear stochastic discrete-time systems (1) and (2), the \(i\)-th rule of the T-S stochastic fuzzy model is set as follows:

\[
R^i: \quad \text{If } z_i(k) \in M_{i1}, \cdots, z_m \in M_{im}, \\
\text{Then} \\
x(k+1) = A_i x(k) + B_i u(k) + w_i(k) 
\]

\[
y(k) = C_i x(k) + v_i(k), i = 1,2,\cdots,r 
\]

Where \(z(k) = [z_1(k), z_2(k), \ldots, z_m(k)]^T\) is the antecedent variable; \(M_{ij}\) is the fuzzy domain; \(r\) is the number of \(If\)-then rules; \(w_i(k), v_i(k)\) and \(x(0)\) are the local uncorrelated process noise, measurement noise and initial state, all of which obey the Gaussian distribution, i.e. \(w_i(k) \sim N(0,\Sigma)\) and \(v_i(k) \sim N(0,\Theta)\); \(A_i, B_i\) and \(C_i\) are the model matrices of the local linear system under the \(i\)-th rule that have appropriate dimensions.

The complex system is locally simplified with \(r\) fuzzy rules. Because the system uncertainty carries clear statistical characteristics, each rule represents the local linear stochastic control problem to be solved by the complex system, making problem \((P)\) a standard LQG problem. Hence, the local control problem under the \(R^i\)-th rule is expressed as:

\[
(LOP) \quad \min_{[u(k)]} E\{J\} \\
\text{s.t.} \quad x(k+1) = A_i x(k) + B_i u(k) + w_i(k), k = 0,1,2,\cdots,N-1 \\
y(k) = C_i x(k) + v_i(k), k = 1,2,\cdots,N 
\]

3. CONTROLLER DESIGN

For the stochastic control problem \((LOP)\) in this research, the objective function is separable, giving the systems the characteristics of linearity. Moreover, the effect of the white noise on the state and measurement is additive. In light of the above, it is possible to resort to Bellman dynamic programming and find the optimal solution through recursion. The following section illustrates the design theory for the control law \(u(k)\) of the T-S stochastic fuzzy model:
Assume that the mean and the variance of the initial state $x(0)$ are $M_0$ and $P_0$, respectively, and the noises $\{w_i(k)\}$ and $\{v_i(k)\}$ are uncorrelated.

**Definition 1**: Based on real-time information $\hat{p}$ at the $k$-th moment, the state $X_i(k)$ of the $i$-th subsystem is estimated to be:

$$\hat{x}_i(k) = E\{x(k) | I^k, i\}$$

Then, $\hat{x}_i(k)$ can be obtained by the Kalman filter recursive equation [1]:

$$\hat{x}_i(k + 1) = A_i \hat{x}_i(k) + B_i u(k) + K_i(k) v_i(k)$$  \hspace{1cm} (6)

Where

$$K_i(k) = A_i P_i(k | k-1) C_i^T [C_i P_i(k | k-1)C_i^T + \Theta_i]^{-1}$$

$$P_i(k + 1 | k) = A_i P_i(k | k) A_i^T + \Omega_i$$

$$P_i(k | k) = [I - K_i(k) C_i] P_i(k | k-1)$$

$$v_i(k) = y(k) - C_i \hat{x}_i(k | k-1)$$

The boundary conditions are $\hat{x}_i(0) = m_0$ and $P(0 | 0) = P_0$.

Since the estimated value of the state is determined by the real-time information sequence $\hat{p}$, the sequence will suffer curse of dimensionality with the increase of $k$. The problem is completely overcome by formula (6), the Kalman filter recursive equation.

**Theorem 1**: Under the $i$-th rule $R_i$, the local optimal control is:

$$u_i^*(k) = -L_i(k) \hat{x}_i(k)$$  \hspace{1cm} (7)

Where

$$L_i(k) = D_i^{-1}(k) B_i^T S_i(k + 1) A_i$$  \hspace{1cm} (8)

$$D_i(k) = B_i^T S_i(k + 1) B_i + Q_2$$  \hspace{1cm} (9)

$$S_i(k) = A_i^T S_i(k + 1) A_i + Q_1 - L_i^T(k) D_i(k) L_i(k)$$  \hspace{1cm} (10)

The boundary condition is $S_i(N) = Q_0$.

**Proof**: Under the $i$-th rule $R_i$, the optimal control of the local control problem can be obtained by dynamic programming.

In the $N-1$ stage, there are:
\[
J(N-1) = E\{x^T(N-1)Q_1x(N-1) + u^T(N-1)Q_2u(N-1)\} + x^T(N)S_1x(N)\} I^{N-1}
\]
\[
= E\{x^T(N-1)Q_1x(N-1) + u^T(N-1)Q_2u(N-1)\}
+ [A_1x(N-1) + B_1u(N-1) + w_1(N-1)]^T S_1(N)\]
\[
	imes [A_1x(N-1) + B_1u(N-1) + w_1(N-1)\} I^{N-1}\}
\]
\[
= E\{x^T(N-1)\} Q_1 + A_1^T S_1(N)A_1 + u^T(N-1)\} Q_2 + B_1^T S_1(N)B_1\} u(N-1)
+ 2u^T(N-1)B_1^T S_1(N)A_1x(N-1)\} I^{N-1}\} + Tr(S_1(N)Q_2)
\]

Where \(Tr(X)\) is the trace of matrix \(X\).

Substitute Kalman filter equation (6) into the above formula (8) to get the following:

\[
J(N-1) = \hat{x}_i^T(N-1)\} Q_1 + A_1^T S_1(N)A_1 + u^T(N-1)\} Q_2 + B_1^T S_1(N)B_1\} u(N-1)
+ 2u^T(N-1)B_1^T S_1(N)A_1\} \hat{x}_i(N-1)
+ Tr(S_1(N)Q_2)
\]

Take the minimum \(J(N-1)\) in relation to \(u(N-1)\), i.e.

\[
\frac{\partial J(N-1)}{\partial u(N-1)} = 0
\]

It can be seen that under the rule \(R^*\), the local optimal control is,

\[
u_i^*(N-1) = -L_i(N-1)\} \hat{x}_i(N-1)
\]

The above formula proves the validity of the conclusion of Theorem 1 at \(k=N-1\).

Substitute the optimal control \(u_i^*(N-1)\) into formula (5.12) to get the optimal index in stage N-1:

\[
J^*(N-1) = \hat{x}_i^T(N-1)S_1(N-1)\} \hat{x}_i(N-1) + Tr(S_1(N)P_i(N \mid N - 1))\]

The rest of the proof is conducted by induction. Suppose the optimal index in stage \(K+1\):

\[
J^*(k + 1) = \hat{x}_i^T(k + 1)S_i(k + 1)\} \hat{x}_i(k + 1) + \sum_{j=k+2}^N Tr(S_i(j)P_i(j \mid j - 1))\]

According to the Bellman dynamic programming equation, the following equation must be minimized in relation to \(u(k)\) in stage \(k\):

\[
J(k) = E\{x^T(k)Q_1x(k) + u^T(k)Q_2u(k) + J^*(k + 1)\} I^k\}
\]

It can be inferred from the definition of real-time information set \(I^k\) that \(I^k \subset I^{k+1}\). The smoothness of the expectation operator is expressed as:

\[
E\{\cdot\} I^k = E\{E\{\cdot\} I^{k+1}\} I^k
\]

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The following equation can be obtained by applying the smoothness of the expectation operator to formula (5.14)

\[ J(k) = E\left[ x^T(k)Q_x(k) + u^T(k)Q_u(k)\hat{x}_i(k+1)S_i(k+1)\hat{x}_i(k+1) | I^k \right] \]

+ \sum_{j=k+2}^{N} \text{Tr}[S_i(j)P_j(j | j - 1)] \tag{15} \]

Substitute Kalman filter equation (6) into the above formula (15) to get the following:

\[ J(k) = E\left[ x^T(k)Q_x(k) + u^T(k)Q_u(k)[A_i\hat{x}_i(k) + B_iu(k) + K_i(k)\nu_i(k)]^T \times S_i(k+1)[A_i\hat{x}_i(k) + B_iu(k) + K_i(k)\nu_i(k)] | I^k \right] \]

+ \sum_{j=k+2}^{N} \text{Tr}[S_i(j)P_j(j | j - 1)] \tag{16} \]

The above formula can be rewritten as:

\[ J(k) = \hat{x}_i^T(k)\left[ Q_i + A_i^T S_i(k+1)A_i + u^T(k)Q_u(k) + B_i^T S_i(k+1)B_i \right] u(k) \]

+ 2u^T(k)B_i^T S_i(k+1)A_i\hat{x}_i(k) \tag{17} \]

Minimize the objective function \( J(k) \) in relation to \( u(k) \), and there is:

\[ u_i^*(k) = -L_i(k)\hat{x}_i(k) \]

Substitute the above \( u_i^*(k) \) into \( J(k) \) to obtain the optimal index in stage \( k \):

\[ J^*(k) = \hat{x}_i^T(k)S_i\hat{x}_i(k) + \sum_{j=k+1}^{N} \text{Tr}[S_i(j)P_j(j | j - 1)] \tag{18} \]

Q.E.D.

Theorem 1 points out the design method for LQG control rate of the sub-system after the linearization by T-S fuzzy model. The Kalman filter recursion process of the subsystem is shown in Figure 1.

Theorem 1 demonstrates that the optimal control \( u_i^*(k) \)corresponding to the \( i-th \) subsystem model consists of two parts: the control gain \( L_i(k) \) and the state estimate \( \hat{x}_i(k) \). The former is determined by formulas (8)–(10). Independent to the measurement or state of the system, the control gain is only related to the local model matrices \( A_i, B_i, C_i \) of the system and the matrices \( Q_0, Q_1, Q_2 \) of performance indices. Thus, it can be calculated and stored off-line in advance. As for the state estimate, it is calculated by formula (6) and correlated to information \( V_i(k) = y(k) - C_i\hat{x}_i(k) \). Obviously, the real-time information \( V_i(k) \) must be determined online by the current measurement \( y(k) \), signifying the severability between the control gain \( L_i(k) \) and state estimate \( \hat{x}_i(k) \). This feature brings great convenience to the engineering application.

Since the complex system is locally linearized by different rules at different operating points, rule \( R_i \) corresponds to one subsystem and the \( r \) rules correspond to \( r \) subsystems. According to Theorem 1, each subsystem model has a specific optimal control. Then, in the whole control process, there are \( r \) optimal controls \( u_{i1}(k), u_{i2}(k), \ldots, u_{ir}(k) \) to choose from at the \( k-th \) moment. The next theorem gives the answer to how to
determine the control law to be applied to the actual system.

\[
L(k) = \sum_{i=1}^{r} \omega_i(k)L_i(k) \tag{19}
\]

\[
\hat{x}(k) = \sum_{i=1}^{r} \omega_i(k)\hat{x}_i(k) \tag{20}
\]

\[
\omega_i(z(k)) = \frac{h_i(z(k))}{\sum_{i=1}^{r} h_i(z(k))} \tag{21}
\]

\[
h_i(z(k)) = \prod_{j=1}^{m} M_{ij}(z_j(k)) \tag{22}
\]

**Theorem 2:** For problem \((P)\), the control applied to the system at the \(k\)-th moment is:

\[
u^*(k) = -L(k)\hat{x}(k)
\]

Where

\[
\nu^*(k) = -L(k)\hat{x}(k)
\]

**Proof:** At the \(k\)-th moment, the complex system is simplified as formulas (4) and (6) under the T-S fuzzy rule \(R^i\). The following conclusion can be drawn from Theorem 1:

\[
R^i: \text{If } z_i(k) \in M_{ii}, \ldots, z_m(k) \in M_{im},
Then \quad u^*_i(k) = -L_i(k)\hat{x}_i(k)
\]

Where \(i=1, 2, \ldots, r, r\) is the number of If-Then rules. In accordance with the principles of single point fuzzification, product reasoning and mean weighted solution fuzzification, the following equations are satisfied at the \(k\)-th moment.

**Figure 1.** The design diagram of subsystem controller
\[ h_i(z(k)) = \prod_{j=1}^{m} M_{ij}(z_j(k)) \]

\[ \omega_i(z(k)) = \frac{h_i(z(k))}{\sum_{j=1}^{r} h_j(z(k))} \]

The resulting \( \omega_i(z(k)) \) stands for the weight of the \( i \) -th subsystem in the entire complex system at the \( k \)-th moment. Hence, the control gain \( L(k) \) and state estimate \( \hat{x}(k) \) at the \( k \)-th moment can be obtained as follows:

\[ L(k) = \sum_{i=1}^{r} \omega_i(k) L_i(k) \]

\[ \hat{x}(k) = \sum_{i=1}^{r} \omega_i(k) \hat{x}_i(k) \]

Q.E.D.

Theorem 2 shows that, if the complex system is simplified are simplified into \( r \) subsystems, each subsystem has an optimal local control gain \( L_i(k) \) and a local state estimate \( \hat{x}_i(k) \) at the \( k \)-th moment. On this basis, the optimal control at the \( k \)-th moment can be deducted by the Parking Distance Control principle, that is, take the sum of the weighted memberships of the \( r \) local gains and state estimates.

4. ALGORITHM SIMULATION

Taking a nonlinear system disturbed by stochastic noise, this section illustrates the specific implementation of the proposed control method. The system model is as follows:

\[ \dot{x}_1(t) = x_2(t) + w(t) \]

\[ \dot{x}_2(t) = \frac{9.8 \sin(x_1(t)) - 0.05 x_2^2 \sin(2x_1(t)) - 0.1 \cos(x_1(t))}{0.667 - 0.1 \cos^2(x_1(t))} + w(t) \]

\[ y_1(t) = x_1(t) + v(t) \]

\[ y_2(t) = x_2(t) + v(t) \]

Where \( x_1(t) \) and \( x_2(t) \) describe the states of the system; \( u(t) \) stands for the control of the system; \( y_1(t) \) and \( y_2(t) \) are system outputs; \( w(t) \) and \( v(t) \) are uncorrelated, incremental process noise and measurement noise, respectively. The \( x_1(t) \) is assumed to vary in the range of \([-1.34, 1.34]\), and \( x_2(t) \) in the range of \([-5, 5]\).

In the above model, it is obvious that the second dynamic equation is a complex nonlinear equation, which makes the controller design extremely difficult. Besides, the system contains state noise \( w(t) \) and measurement noise \( v(t) \). In light of prior information, it can be assumed that the system is running within a certain range, e.g. the rectangular shown in Figure 2.
For the sake of simplicity, 9 points are selected from the rectangular area: \((0, 0), (0, \pm 4), (1.05, \pm 4), (\pm 1.05, 0)\) and \((-1.05, \pm 4)\). Following the T-S fuzzy rules, the nonlinear system is locally linearized at the 9 points. Five fuzzy rules are generated, each corresponds to a linear subsystem, i.e.

\[
R^1: \text{ If } z_1(k) \in M_{11} \text{ and } z_2(k) \in M_{12}, \quad \text{Then } \begin{align*} x(k+1) &= A_1x(k) + B_1u(k) + w_1(k) \\ y(k) &= C_1(k) + v_1(k) \end{align*}
\]

\[
R^2: \text{ If } z_1(k) \in M_{21} \text{ and } z_2(k) \in M_{22}, \quad \text{Then } \begin{align*} x(k+1) &= A_2x(k) + B_2u(k) + w_2(k) \\ y(k) &= C_2(k) + v_2(k) \end{align*}
\]

\[
R^3: \text{ If } z_1(k) \in M_{31} \text{ and } z_2(k) \in M_{32}, \quad \text{Then } \begin{align*} x(k+1) &= A_3x(k) + B_3u(k) + w_3(k) \\ y(k) &= C_3(k) + v_3(k) \end{align*}
\]

\[
R^4: \text{ If } z_1(k) \in M_{41} \text{ and } z_2(k) \in M_{42}, \quad \text{Then } \begin{align*} x(k+1) &= A_4x(k) + B_4u(k) + w_4(k) \\ y(k) &= C_4(k) + v_4(k) \end{align*}
\]

\[
R^5: \text{ If } z_1(k) \in M_{51} \text{ and } z_2(k) \in M_{52}, \quad \text{Then } \begin{align*} x(k+1) &= A_5x(k) + B_5u(k) + w_5(k) \\ y(k) &= C_5(k) + v_5(k) \end{align*}
\]

Where analytic expression of the membership function \(M_{ij}\) in the \(i\)-th rule \(R^i\) is as follows:

\[
M_{11} = e^{-\frac{(z_1-0)^2}{2\times0.3^2}}, \quad M_{12} = e^{-\frac{(z_2-0)^2}{2\times1.2^2}}
\]
\[ M_{21} = e^{-\frac{(z_1-0)^2}{2\sigma_1^2}}, M_{22} = e^{-\frac{(z_2-4)^2}{2\sigma_2^2}} \]

\[ M_{31} = e^{-\frac{(z_1+0.5)^2}{2\sigma_1^2}}, M_{32} = e^{-\frac{(z_2-0)^2}{2\sigma_2^2}} \]

\[ M_{41} = e^{-\frac{(z_1-1.05)^2}{2\sigma_1^2}}, M_{42} = e^{-\frac{(z_2-4)^2}{2\sigma_2^2}}, or, M_{41} = e^{-\frac{(z_1+1.05)^2}{2\sigma_1^2}}, M_{42} = e^{-\frac{(z_2-4)^2}{2\sigma_2^2}} \]

\[ M_{51} = e^{-\frac{(z_1-0.5)^2}{2\sigma_1^2}}, M_{52} = e^{-\frac{(z_2+4)^2}{2\sigma_2^2}}, or, M_{51} = e^{-\frac{(z_1+0.5)^2}{2\sigma_1^2}}, M_{52} = e^{-\frac{(z_2+4)^2}{2\sigma_2^2}} \]

Set the system sampling time \( T=0.1 \) seconds, the corresponding discrete subsystem model matrix is as follows:

\[ A_1 = \begin{bmatrix} 1.088 & 1.780 \\ 0.103 & 1.088 \end{bmatrix}, B_1 = \begin{bmatrix} -0.016 \\ -0.018 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ A_2 = \begin{bmatrix} 1.073 & 1.490 \\ 0.102 & 1.073 \end{bmatrix}, B_2 = \begin{bmatrix} -0.013 \\ -0.018 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ A_3 = \begin{bmatrix} 1.029 & 0.891 \\ 0.101 & 1.029 \end{bmatrix}, B_3 = \begin{bmatrix} -0.002 \\ -0.008 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ A_4 = \begin{bmatrix} 1.037 & 0.753 \\ 0.104 & 1.093 \end{bmatrix}, B_4 = \begin{bmatrix} -0.003 \\ -0.008 \end{bmatrix}, C_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ A_5 = \begin{bmatrix} 1.036 & 0.714 \\ 0.099 & 0.983 \end{bmatrix}, B_5 = \begin{bmatrix} -0.003 \\ -0.008 \end{bmatrix}, C_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

**Figure 3.** The membership functions of state vectors \( x_1 \) and \( x_2 \)

At the \( k \)-th moment, the state variables \( x_1(k) \) and \( x_2(k) \) of the system are unpredictable, but it is possible to make estimation of the state variables \( \hat{x}_1(k) \) and \( \hat{x}_2(k) \) with known information \( F \). The local linear model corresponding to each rule can be fully determined by replacing the state variables \( x_1(k) \) and \( x_2(k) \) with variables
\( \hat{x}_1(k) \) and \( \hat{x}_2(k) \) in antecedent variables \( z_1(k) \) and \( z_2(k) \).

Figure 3 shows the shape of the membership functions of the state vectors \( x_1(k) \) and \( x_2(k) \), which are exactly the same as the membership functions in other references. Therefore, this paper also assumes that the membership functions obey normal distribution. In light of the unpredictability, \( x_1(k) \) and \( x_2(k) \) are substituted by the state estimates \( \hat{x}_1(k) \) and \( \hat{x}_2(k) \).

Suppose the noise in each local model of the system has exactly the same statistical characteristics, i.e.

\[
\begin{align*}
\nu_1(k) & \sim N(0,0.8) \\
\nu_2(k) & \sim N(0,0.1)
\end{align*}
\]

The mean and variance of the noise in each local model will remain unchanged. In contrast, if the local models have different statistical characteristics, there will be only slight changes to the values of statistical characteristics while the controller design method will be exactly the same. With the design method for nonlinear stochastic fuzzy controller in Section 3, the system’s simulation results can be obtained (Figure 4) on the assumption that the initial state of the system is a random number in the rectangle shown in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{response_curves}
\caption{The response curves of state vectors \((x_1, x_2)\)}
\end{figure}

Figure 4 displays the state response curves of the system. The “green dotted lines” illustrate the target value of the system state, while the “blue solid lines” depict the actual response curves of the system in the control strategy designed in Section 3. The curves demonstrate that: after adopting the proposed method, the state vectors of the nonlinear system quickly converge to the desired position under the action of the control vector \( u \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{control_vector_response}
\caption{The response curve of control vector \((u)\)}
\end{figure}
Figure 5 is the curve graph of the control vector $u$, which tends to zero within the finite number of steps. This means the desired control is achieved with minimal control energy.

5. CONCLUSION

Targeted at the optimal control of nonlinear stochastic discrete-time systems disturbed by additive noise under the quadratic performance index, this paper designs a controller to regulate nonlinear stochastic systems under the framework of the T-S fuzzy control. The proposed controller is proved to be effective by the system simulation results. Thanks to the severability between state estimation and control gain, the proposed controller design method is easy to calculate and implement. Suffice it to say that this research provides an effective controller design plan to highly nonlinear stochastic models in aircraft control.

REFERENCES