Discrete Strategies for Multiplayer Games Using Quantum Formalism

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Abstract
Although lots of solutions such as bounded rationality theory or expected utility hypothesis seem like the best models to describe the process of strategies, they are difficult to make clear the corresponding violations, irrationalities or paradoxes. Recently the perspective of quantum algorithms that can take into account the above difficulties has been introduced into the process. In this paper we attempt to expand the quantum formalism of the prisoners’ dilemma (two-player and two-choice, i.e., 2×2 games) to the N×2 and N×M games for N non-communicating players who have to choose the unique one between a series of discrete choices. For N×2 games with the assistance of entanglement and linear superposition, initial states with different level of entanglement are discussed. For N×M games using three entangled qutrits as an example, the initial state is expanded to a group of GHZ-type states. It is shown that the players individually prefer to pay high attention to the collective interests gradually.

Key words: Prisoners’ Dilemma, Entanglement, Linear Superposition.

1. INTRODUCTION

Game theory, as a new branch of modern mathematics and a very important subject on operational research, has now found a wide range of applications in multitudinous fields such as computer science, economics, medical science, biology, etc. For a long time, approaches such as expected utility theory (Gilboa and Schmeidler, 1989; Quiggin, 1995; Miyamoto and Wakker, 1996), rational choice theory (Calvert, 1995; Hechter and Kanazawa, 1997; Kiser and Hechter, 1998), and multi-criteria decision making (Tarp and Helles, 1995; Gogus, 1997; Triantaphyllou and Evans, 1999) seemed like the mainstream models to explain, analyze and predict the process of a game. However, people do not at all comply with the regulations of rational choice theory and expected utility hypothesis: they are almost bounded rational and their choices’ probabilistic. There have been various methods of bounded rationality such as systematic errors (Arkes and Hammond, 1985; Nisbett and Ross, 1980; Pitz and Sachs, 1984), heuristics or rules of thumb (Ho, 1994; Ballou, 1989; Geigerenzer and Selten, 2001), stochastic choice rules (Thurstone, 1927; Luce, 1959; Blavatskyy, 2012), and quantal response equilibrium (Bajari and Hortacsu, 2005; Camerer, Palfrey, and Rogers, 2009; Golman, 2011). Unfortunately, all of these mentioned ways could only rock a subset of the paradoxes. The literatures above cannot escape the common main modeling approach, such as, mixed integer linear programming model, the exponential distribution or integral model under various parameters, etc. That is, most researches are based on the classical probability theory whose decision space is finite, limited and smaller. However, quantum mechanics has a bigger probability space and brings a totally new perspective. Quantum theory has created an invigorating buzz for the academic and is attracting new visitors in large. Many other fields like large-scale calculations, nuclear energy and aerospace, etc., have combined with it, resulting in a range of new interdisciplinary sciences.

An increasing number of scholars have been into the interdisciplinary science of quantum strategies since 1999. However most work in the beginning was done in studying two person games. Meyer considered game theory from the perspective of quantum algorithms, showing the power of quantum strategies for the first time (Meyer, 1999). Eisert et al. introduced an elegant scheme for quantizing classical games, and proceeded to perform an extensive analysis of its application to the famous two-player game, Prisoner’s Dilemma (Eisert, Wilkens and Lewenstein, 1998). After that, Marinatto et al. extended the concept of a classical two-person static game to the quantum domain, by giving a Hilbert structure to the space of classical strategies and studying the Battle of the Sexes game (Marinatto and Weber, 2000). Chen et al. showed that if the handicapped player with classical means can delay his action for a sufficiently long time, the quantum version reverts to the classical zero-sum game under decoherence (Chen, Kwek and Oh, 2002). Frąckiewicz presented the unique solution to the quantum Battle of the Sexes game, showing the best result to be achieved when the game is played according to Marinatto and Weber's scheme (Frąckiewicz, 2009). Frąckiewicz gave a strict mathematical description for a refinement of the Marinatto–Weber quantum game scheme. The model allows the players to choose projector operators that determine the state on which they perform their local operators (Frąckiewicz, 2014). Few of many others who contributed to the study of multiplayer games are given. Benjamin and Hayden were the first to study multiplayer games (Benjamin and Hayden, 2000). Khan et al. presented a quantum model
of Bertrand duopoly and studied the entanglement behavior on the profit functions of the firms. Using the concept of optimal response of each firm to the price of the opponent, they found only one Nash equilibrium point for the maximally entangled initial state (Khan and Ramzan, 2010). Du et al. investigated the 3-player quantum Prisoner's Dilemma with a certain strategic space, a particular Nash equilibrium that can remove the original dilemma is found. Based on this equilibrium, they showed that the game is enhanced by the entanglement of its initial state (Du, Hui, Xu, Zhou, and Han, 2002). Li et al. investigated the quantization of games in which the players can access to a continuous set of classical strategies, making use of continuous-variable quantum systems (Li, Du, and Massar, 2003). Ma et al. constructed a quantum structure where each firm is represented by a qubit from the extended model of the Cournot’s duopoly and then they let these firms entangle pairwise (Ma, Zhou, and Li, 2006). However most work in the study of multiplayer games was discussed in terms of space-continuous situation: the strategies are continuous, rather than discrete.

To answer the discrete situation of multiplayer games, we develop a general model of quantum formalism that takes into account the following two cases: (i) binary strategy process, due to a player’s search process for the only best answer from just two alternatives; and/or (ii) non-binary strategy process, where a set of players have to choose the best one from m discrete choices. In this paper we therefore attempt to expand the quantum formalism of the prisoners’ dilemma (two-player and two-choice, i.e., 2×2 games) to the N×2 (n-player and two-choice) and N×M (n-player and m-choice) games.

2. PRELIMINARIES AND PROTOCOL

In this section, we focus our attention on the basic mathematical formalism describing the quantization of games.

2.1. The Classic Context

In general, a game is defined as a set

\[ \Gamma = \{N, \{s_i\}, \{S_i\}\} \]  

(1)

where \(N\) denotes the number of players, \(\{s_i\}\) denotes the set of available strategies of player \(i\), and \(\{S_i\}\) denotes the payoffs of relevant outcomes. Then, the strategy space would be

\[ S = S_1 \times S_2 \times \cdots \times S_n \]  

(2)

Then we have its payoff functions \$s_i$

\[ S_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R} \]  

(3)

Let \(\Delta(S_i)\) be the set of convex linear combinations of the elements \(s_i' \in S_i\). The mixed strategy \(s_{mix} \in \Delta(S_i)\) is then given by:

\[ \sum_{i \in N} p_i s_i' \quad \text{with} \quad \sum_{i} p_i = 1 \]  

(4)

where \(p_i\) is the probability player \(i\) assigns to the choice \(s_i'\). The space of mixed strategies would be

\[ \Delta(S) = \Delta(S_1) \times \Delta(S_2) \times \cdots \times \Delta(S_n) \]  

(5)

which contains all possible mixed strategy profiles \(\sigma_{mix}\). We now have:

\[ S_i : \Delta(S_1) \times \Delta(S_2) \times \cdots \times \Delta(S_n) \to \mathbb{R} \]  

(6)

2.2 The Quantum Concepts

We introduce the basic definitions and concepts necessary to utilize the mathematical framework of quantum mechanics.

**Hilbert Spaces and Dirac’s Notation**

For two elements \(u\) and \(v\) of an \(n\)-dimensional complex vector space \(\mathbb{C}^n\), the map \(\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}\) given by

\[ \langle u, v \rangle = v^T I u \]  

(7)

where \(I\) is the \(n \times n\) identity matrix, defines an inner product in \(\mathbb{C}^n\). The associated norm \(\| \cdot \|\) is given by

\[ \| u \| = \sqrt{\langle u, u \rangle} \]  

(8)

A general \(n\) qubit system can therefore be written

\[ |\psi\rangle = \sum_{s_{1},\ldots,s_{n} = 0}^{1} a_{s_{1},\ldots,s_{n}} |s_{1} \cdots s_{n}\rangle \]  

(9)

where
\[ |x_1 \cdots x_n \rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle \in \mathbb{H}^n \]  
(10)
with \( x_i \in \{0,1\} \) and complex coefficients \( a_{x_i} \). For a two qubit system, we have
\[ |\psi\rangle = \sum_{x_1,x_2} a_{x_1,x_2} |x_1x_2\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle \]  
(11)
with \( x_i \in \{0,1\} \) and complex coefficients \( a_{x_i} \). For a two qubit system, we have

**Axioms**

A qubit is a representation of a two-level quantum state, which lives in a two dimensional complex space spanned by two basis states denoted \( |0\rangle \) and \( |1\rangle \), corresponding to the two states of the classical bit. The qubit can be in any superposition of \( |0\rangle \) and \( |1\rangle \) according to Bloch-sphere:
\[ |\psi\rangle = a_0|0\rangle + a_1|1\rangle \]  
(13)
where \( a_0 \) and \( a_1 \) are arbitrary complex numbers satisfying \( a_0^*a_0 + a_1^*a_1 = 1 \) for normalization, meanwhile \( |a_0|^2 + |a_1|^2 = 1 \). \( |a_i|^2 \) is simply the probability to find the system in the state \( |i\rangle \), \( i \in \{0,1\} \). The state of an arbitrary qubit can be written in the computational basis as:
\[ |\psi\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \]  
(14)
and can get:
\[ |\psi\rangle = \cos \left( \frac{\theta}{2} \right) |0\rangle + e^{i\phi} \sin \left( \frac{\theta}{2} \right) |1\rangle = \cos \left( \frac{\theta}{2} \right) |0\rangle + (\cos \phi + i\sin \phi) \sin \left( \frac{\theta}{2} \right) |1\rangle \]  
(15)
where \( 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi \), \( \hat{x} = \sin \theta \cos \phi \), \( \hat{y} = \sin \theta \sin \phi \), \( \hat{z} = \cos \theta \), the parameters \( \theta \) and \( \phi \) are re-interpreted in spherical coordinates, specifying a point. \( \hat{a} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), on the unit sphere \( \mathbb{S}^2 \).

![Fig. 1 Bloch sphere](image)

**Entanglement**

An entangled state is basically a quantum system that cannot be written as a tensor product of its subsystems, we’ll thus define two classes of quantum states. Examples below refers to two-qubit states.

**Product states:**
\[ |\psi_0\rangle = |\psi_{0_1}\rangle \otimes |\psi_{0_2}\rangle \]  
(16)
and entangled states
\[ |\psi_0\rangle \neq |\psi_{0_1}\rangle \otimes |\psi_{0_2}\rangle \]  
(17)
For a mixed state, the density matrix is defined as mentioned by \( \rho_0 = \sum_i p_i |\psi_i\rangle \langle \psi_i| \) and it is said to be separable, which we will denote by \( \rho_0^{sep} \), if it can be written as
\[ \rho_0^{sep} = \sum_i p_i (\rho_{0_1} \otimes \rho_{0_2}) \]  
(18)
A set of very important two-qubit entangled states are the Bell states
\[ |\Phi^+_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \quad |\Psi^+_0\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \]  
(19)
The GHZ-type-states
\[ |\text{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|00\ldots 0\rangle + e^{i\gamma}|11\ldots 1\rangle) \] (20)

could be seen as a \( n\)-qubit generalization of \( |\Phi^+\rangle \)-states.

**Physical operations on joint states**

We can create a projector \( P \), by taking the outer product of a vector with itself:

\[ P = |\phi\rangle \langle \phi| \] (21)

\( P \) is a matrix with every element \( P_{ij} \) being the product of the elements \( i, j \) of the vectors in the outer product. This operator projects any vector \( |\gamma\rangle \) onto the 1-dimensional subspace of \( \mathbb{1} \), spanned by \( |\phi\rangle \):

\[ P|\gamma\rangle = |\phi\rangle \langle \phi| |\gamma\rangle = \langle \phi|\gamma\rangle |\phi\rangle \] (22)

It simply gives the portion of \( |\gamma\rangle \) along \( |\phi\rangle \). The generators of the group are the Pauli spin matrices \( \sigma_x, \sigma_y, \sigma_z \) shown together with the identity matrix \( I \):

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & -i \\
i & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

(23)

Note that \( \sigma_x \) is identical to a classical (bit-flip) \( \text{NOT} \)-operation. General \( 2 \times 2 \) unitary operators can be parameterized with three parameters \( \theta, \alpha, \beta \), as follows:

\[
U(\theta, \alpha, \beta) = \begin{bmatrix}
e^{i\alpha} \cos \frac{\theta}{2} & ie^{i\beta} \sin \frac{\theta}{2} \\
-ie^{-i\beta} \sin \frac{\theta}{2} & e^{-i\alpha} \cos \frac{\theta}{2}
\end{bmatrix}
\]

(24)

For a general \( n\)-qubit \( |\psi\rangle \) we get:

\[
U_1 \otimes U_2 \otimes \cdots \otimes U_n |\psi\rangle = \sum_{a_1, a_2, \ldots, a_n} a_{a_1 a_2 \ldots a_n} U_{a_1} |x_1\rangle \otimes U_{a_2} |x_2\rangle \otimes \cdots \otimes U_n |x_n\rangle
\]

(25)

We have so far only discussed pure states. So for example a state that is in \( |\psi\rangle = a_1 |0\rangle + a_2 |1\rangle \) with probability \( p_1 \) and in \( |\psi_2\rangle = a_3 |0\rangle + a_4 |1\rangle \) with probability \( p_2 \) is mixed. We handle mixed states by defining a density operator \( \rho \), which is a Hermitian matrix with unit trace:

\[
\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|
\]

(26)

where \( \sum_i p_i = 1 \). If we apply a unitary operator \( U \) on a pure state, we end up with \( U |\psi\rangle \) which has the density operator \( U \rho U^+ = U |\psi\rangle \langle \psi| U^+ \).

### 2.3. The Protocol for 2 × 2 Games

In Eisert’s protocol (Eisert, Wilkens and Lewenstein, 1998) the qubits are prepared in an entangled state by applying to an initial state \( |\psi_\text{in}\rangle = |00\rangle \) the entangling operator

\[
J(\gamma) = \cos \left(\frac{\gamma}{2}\right) I \otimes I + i \sin \left(\frac{\gamma}{2}\right) X \otimes X
\]

\[
= \begin{bmatrix}
\cos \left(\frac{\gamma}{2}\right) & 0 & 0 & i \sin \left(\frac{\gamma}{2}\right) \\
0 & \cos \left(\frac{\gamma}{2}\right) & i \sin \left(\frac{\gamma}{2}\right) & 0 \\
0 & i \sin \left(\frac{\gamma}{2}\right) & \cos \left(\frac{\gamma}{2}\right) & 0 \\
i \sin \left(\frac{\gamma}{2}\right) & 0 & 0 & \cos \left(\frac{\gamma}{2}\right)
\end{bmatrix}
\]

(27)

where \( I \) is the single qubit identity operator, \( X \) is the bit-flip operator (8), and \( \gamma \in [0, \pi/2] \) is a measure of the degree of entanglement. Maximal entanglement is achieved by setting \( \gamma = \pi/2 \), since

\[
J(\pi/2) |00\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + i |11\rangle \right)
\]

(28)
The game proceeds by Alice and Bob performing their local strategies $U_A$ and $U_B$, and the state is turned into its final form

$$|\psi_{fin}\rangle = (U_B \otimes U_A) |00\rangle$$  \hspace{1cm} (29)

It is considered a two parameter subset of $SU(2)$ as the strategy space:

$$U(\theta, \alpha) = \begin{pmatrix} e^{i\alpha} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & e^{-i\alpha} \cos(\theta/2) \end{pmatrix}$$  \hspace{1cm} (30)

Instead a new Nash equilibrium emerges at

$$U_A = U_B = U(0, \pi/2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$  \hspace{1cm} (31)

### 2.4. General Considerations for Multiplayers

The multi-player protocol is a natural extension of the two-player one. The initial state is now $N$ qubits $|00\cdots0\rangle$ entangled by a generalization of

$$J(Y) = \cos\left(\frac{Y}{2}\right) J^N + i\sin\left(\frac{Y}{2}\right) X^N$$  \hspace{1cm} (32)

to produce the $N$-qubit Greenberger-Horne-Zeilinger (GHZ) state

$$\frac{1}{\sqrt{2}} \left(|00\cdots0\rangle + i|11\cdots1\rangle\right)$$  \hspace{1cm} (33)

The final game state is computed by

$$\rho_{\text{fin}} = U \otimes U \cdots \otimes U |\rho_i\rangle^N \otimes U^* \cdots \otimes U^*$$  \hspace{1cm} (34)

We have the expected payoffs $E(S)$:

$$P_i = \sum_j S_j \langle \chi_j | \chi_i \rangle$$  \hspace{1cm} (35)

where the states $|\chi_i\rangle$ are those states that lead to a payoff for player $i$, and $S_j$ the associated payoffs. The expected payoff $E(S_i)$ of player $i$ is calculated by taking the trace of the product of the final state $\rho_{\text{fin}}$ and the payoff-operator $P_i$:

$$E_i(S) = \text{Tr}(P_i \rho_{\text{fin}})$$  \hspace{1cm} (36)

### 3. Binary Quantum Strategy Process

In a one-shot game the players can do no better than selecting the mixed strategy of choosing each of the two options with probability $1/2$. The payoff to each player is then

$$\langle S \rangle = \sum_{i=1}^{\binom{N+1}{2}} S_i C_i \frac{2^k (1 \wedge_i^N)}{N(N-1)/2}$$  \hspace{1cm} (37)

The NE strategy for the four player game is

$$\hat{s}_{\text{NE}} = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{16}\right) (i + i\hat{X}) - \frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{16}\right) (i\hat{Y} + i\hat{Z})$$

$$= M\left(\frac{\pi}{2}, -\frac{\pi}{16}\right)$$  \hspace{1cm} (38)

A four qubit version of this state is

$$|\Psi(x)\rangle = \frac{x}{\sqrt{2}} |\text{GHZ}_x\rangle + \frac{1-x^2}{2^{N/2}} (|01\rangle + |10\rangle)^{\otimes N/2}$$  \hspace{1cm} (39)

Then, we form a density matrix $\rho_x$ out of $|\Psi(x)\rangle$ and add noise that can be controlled by the parameter $f$. We get:
\[
\rho_n = f |\Psi_n\rangle \langle \Psi_n | + \frac{1-f}{64} I_{64}
\] (40)

where \( I_{64} \) is the 64 \( \times \) 64 identity matrix. For the GHZ-state alone i.e. \( x = 1 \) and \( f = 1 \) it has been shown that the Nash equilibrium solution \( s_{N} = M(\theta, \alpha, -\alpha) \) for the 4-player game is \( M\left(\frac{\pi}{2}, \frac{\pi}{8}, \frac{\pi}{8}\right) \), and for the 6-player game, \( M\left(\frac{\pi}{2}, \frac{\pi}{12}, \frac{\pi}{12}\right) \). For \( f = 1 \), the payoff is given by

\[
\langle S \rangle = \frac{1}{4} + \frac{x^2}{16}
\] (41)

When noise is taken into account the payoff function becomes

\[
\langle S \rangle = \frac{1}{16}(3 + f + f\theta^2)
\] (42)

When the noise reaches maximum: \( f \to 0 \), the classical payoff of 3/16 returns. The following inequality holds for a Nash equilibrium:

\[
S_i(M^m_{\alpha}) \geq S_i(M_{\beta} \otimes M^m_{\alpha})
\] (43)

A GHZ-state can be created by acting with an entanglement operator \( J(\gamma) \) on a product state \([0] \otimes [0] \otimes \cdots \otimes [0]\), where

\[
J(\gamma) = \exp\left(\frac{i2\gamma}{2} \sigma^{xy}\right)
\] (44)

We then have

\[
|\Psi(\gamma)\rangle = |00\cdots 0\rangle
\] (45)

where \( \gamma \in \left[0, \frac{\pi}{2}\right] \) is a parameter that controls the level of entanglement. This gives an output state of the following form

\[
|\Psi(\gamma)\rangle = \cos\left(\frac{\gamma}{2}\right)|00\cdots 0\rangle + i\sin\left(\frac{\gamma}{2}\right)|11\cdots 1\rangle
\] (46)

A \( N \)-player generalization has been conjectured:

\[
\langle S \rangle_N = \langle S \rangle_c - \frac{1}{2}\langle S \rangle_{01} - \frac{1}{2}\langle S \rangle_{02} + \frac{1}{2}\langle S \rangle_{03} - \frac{1}{2}\langle S \rangle_{10} - \frac{1}{2}\langle S \rangle_{12} + \frac{1}{2}\langle S \rangle_{20} - \frac{1}{2}\langle S \rangle_{21} + \frac{1}{2}\langle S \rangle_{23}
\] (47)

where \( \langle S \rangle_c \) is the classically obtainable payoffs for classical NE strategies. A six player game could use a product of two three qubit W-states as its initial state \(|\Psi_n\rangle\).

\[
|\Psi_n\rangle = |W_3\rangle \otimes |W_3\rangle
\] (48)

where

\[
|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)
\] (49)

|\Psi_n\rangle \) is a symmetric superposition of nine states with four qubits in the \([0]\) -state and two in the \([1]\) -state, compactly written as \(|\Psi_n\rangle = |4, 2\rangle\).

### 4. Non-Binary Quantum Strategy Process

A qutrit is a 3-level quantum system on 3-dimensional Hilbert space \( \mathbb{C}^3 \), written in the computational basis as:

\[
|\Psi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle \in \mathbb{C}^3
\] (50)

with \( a_0, a_1, a_2 \in \mathbb{C} \) and \( |a_0|^2 + |a_1|^2 + |a_2|^2 = 1 \). A general \( n \)-qutrit system \(|\Psi\rangle\) is a vector on \( 3^n \)-dimensional Hilbert space, and is written as a linear combination of \( 3^n \) orthonormal basis vectors.

\[
|\Psi\rangle = \sum_{x_0, x_1, \ldots, x_n} a_{x_0, x_1, \ldots, x_n} |x_0 \cdots x_n\rangle
\] (51)

where

\[
|x_0 \cdots x_n\rangle = |x_0\rangle \otimes |x_1\rangle \otimes \cdots \otimes |x_n\rangle \in \mathbb{C}^3 \otimes \cdots \otimes \mathbb{C}^3
\] (52)
with $x_i \in \{0,1,2\}$ and complex coefficients $a_{x_i}$, obeying $\sum |a_{x_i}|^2 = 1$.

Single qutrits are transformed with unitary operators $U \in SU(3)$, i.e., operators from the special unitary group of dimension 3, acting on $\mathbb{C}^3$ as $U : \mathbb{C}^3 \rightarrow \mathbb{C}^3$. The SU(3) matrix is parameterized by defining three general, mutually orthogonal complex unit vectors $\pi$, $\bar{\pi}$, $\bar{\pi}'$, such that $\pi \cdot \bar{\pi} = 0$ and $\pi' \cdot \bar{\pi}' = \pi'$. Now a general complex unit vector is given by:

$$\pi = \begin{pmatrix} \sin \theta \cos \phi e^{a_i} \\ \sin \theta \sin \phi e^{a_i} \\ \cos \theta e^{a_i} \end{pmatrix}$$

(53)

and one complex unit vector orthogonal to $\pi$ is given by:

$$\bar{\pi} = \begin{pmatrix} \cos \chi \cos \theta e^{\beta_0-a_i} + \sin \chi \sin \phi e^{\beta_0-a_i} \\ - \cos \chi \cos \theta e^{\beta_0-a_i} - \sin \chi \sin \phi e^{\beta_0-a_i} \\ - \cos \chi \sin \theta e^{\beta_0-a_i} \end{pmatrix}$$

(54)

where $0 \leq \phi, \chi, \theta \leq \pi / 2$ and $0 \leq \alpha, \alpha_2, \alpha_3, \beta_1, \beta_2 \leq 2\pi$. We have a general SU(3) matrix $U$, given by:

$$U = \begin{pmatrix} x_1 y_3 y_1 y_2 y_1 - y_1 y_3 y_1 \\ x_2 y_3 y_1 y_2 y_1 - y_1 y_3 y_1 \\ x_3 y_3 y_1 y_2 y_1 - y_1 y_3 y_1 \end{pmatrix}$$

(55)

and it is controlled by eight real parameters $\phi, \theta, \chi, \alpha, \alpha_2, \alpha_3, \beta_1, \beta_2$.

The initial state, a maximally entangled GHZ-type state

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$

(56)

is symmetric and unbiased in regards to permutation of player position and has the property of letting us embed the classical version of the game, accessible trough restrictions on the strategy sets. The cyclic group of order three, $C_3$, generated by the matrix:

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(57)

where $s^1 = s^0 = I$ and $s^2 = s^1$. The set of classical strategies $S = \{s^i, s^i, s^1\}$ with $s^i \otimes s^i \otimes s^i |000\rangle = |ijk\rangle$ acts on the initial state $|\psi_{\text{in}}\rangle$ as:

$$s^i \otimes s^i \otimes s^i \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) =$$

$$\frac{1}{\sqrt{3}}(|0 + i0 + j0 + k\rangle + |i + i0 + j1 + k\rangle + |2 + i2 + j2 + k\rangle)$$

(58)

Note that the superscripts denote powers of the generator and that the addition is modulo 3. In the case under study, where there is no preference profile over the different choices, any combination of the operators in $S = \{s^i, s^i, s^1\}$ leads to the same payoffs when applied to $|\psi_{\text{in}}\rangle$ as to $|000\rangle$. We form a density matrix $\rho_{\text{in}}$ out of the initial state $|\psi_{\text{in}}\rangle$ and add noise that can be controlled by the parameter $f$. We get:

$$\rho_{\text{in}} = f|\psi_{\text{in}}\rangle\langle \psi_{\text{in}}| + \frac{1-f}{27}I_27$$

(59)

where $I_27$ is the $27 \times 27$ identity matrix. Alice, Bob and Charlie now applies a unitary operator $U$ that maximizes their chances of receiving a payoff $S = 1$, and thereby the initial state $\rho_{\text{in}}$ is transformed into the final state $\rho_{\text{fin}}$:

$$\rho_{\text{fin}} = U \otimes U \otimes U \rho_{\text{in}} U^\dagger \otimes U^\dagger \otimes U^\dagger$$

(60)

We define for each player $i$ a payoff-operator $P_i$, which contains the sum of orthogonal projectors associated with the states for which player $i$ receives a payoff $S = 1$. For Alice this would correspond to

$$P_A = \sum_{x_1, y_1, y_2 \neq x_1} |\psi_{\text{in}}\rangle \langle \psi_{\text{in}}| +$$

$$+ \sum_{x_1, y_1, y_2 \neq x_1} |\psi_{\text{in}}\rangle \langle \psi_{\text{in}}|$$

(61)

The expected payoff $E(S)$ of player $i$ is as usual calculated by taking the trace of the product of the final state $\rho_{\text{fin}}$ and the payoff-operator $P_i$:

$$E(S) = Tr(P_i \rho_{\text{fin}})$$

(62)
5. CONCLUSIONS

In this paper we attempt to expand the quantum formalism of the prisoners’ dilemma to the N×2 and N×M games for N non-communicating players who have to choose the unique one between a series of discrete choices. For N×2 games with the assistance of entanglement and linear superposition, initial states with different level of entanglement are discussed. For N×M games using three entangled qutrits as an example, the initial state is expanded to a group of GHZ-type states. It is shown that the players individually prefer to pay high attention to the collective interests gradually. The achievement of this performance is highly dependent on the strength of entanglement and the fidelity of the initial state.

REFERENCES


