Fractional Order Multivariable Grey Model Based on Discrete Exponential Function Optimization

Qiyuan Liu*
1. State key Laboratory of Advanced Design and Manufacture for Vehicle Body, Hunan University, Changsha, Hunan, 410082, P.R. China
2. College of Mechanical Engineering, Hunan University of Arts and Science, Changde, 415000, P.R. China
Dejie Yu
State key Laboratory of Advanced Design and Manufacture for Vehicle Body, Hunan University, Changsha, Hunan, 410082, P.R. China
*Corresponding author(E-mail:lqy670@163.com)

Abstract
Background value structure is one of the most effective methods to improve modeling accuracy of grey model. This paper firstly analyzed fractional order accumulated generation operation (AGO) and inverse accumulated generation operation (IAGO) of single variable. Then, formula of multivariable AGO was deduced based on discrete exponential function optimization. After that, the multivariable grey model FMGM (1, n) with fractional order accumulation was established, and the model parameter estimation was derived by least square method. By taking fractional order and minimum average relative error as design variable and object function, this paper established optimal model and wrote solution program in MATLAB. As natural promotion of single variable model FGM (1, 1), multivariable grey model FMGM (1, n) with fractional order accumulation reflected the interaction of variables. At last, the numerical examples were given to indicate correctness and effectiveness of the model.

Key words: Fractional Order, Grey Model, Exponential Function, Least Square Method, Model Parameter Estimation.

1. INTRODUCTION
As one of grey system theories, GM (1, 1) model was widely used in fields such as industry, society and economy. GM (1, 1) was the most common grey model. It applied first order differential equation model (integer order derivative model) of single variable to reveal inherent development law in modeling and prediction of single time series. In actual system, fractional order was used to explore essential characteristics and behaviors of objects with fractional order properties. Based on grey model GM (1, 1), the grey model FGM (1, 1) with fractional order accumulation was proposed (Wu and Liu, 2013). FGM (1, 1) was applied to predict weapon maintenance cost (Fang and Wu, 2013). The discrete grey model DFGM (1, 1) with fractional order accumulation was discussed (Wu and Liu, 2014). The optimal solution of grey model with fractional order accumulation was discussed (Shen and Qin, 2014). Actual social and economic systems consisted of correlative variables. These variables mutually affected each other. The single variable model GM (1, 1) was developed to multivariable model MGM (1, n) (Zhai and Sheng, 1997). In MGM (1, n), background value was generated by mean. Grey model MGM (1, n) is not simple combination of GM (1, 1). Different from GM (1, n), MGM (1, n) established n n-element differential equations for solution. Therefore, parameters in MGM (1, n) reflected interaction of variables. Background value structure is one of the most effective methods to improve modeling accuracy of grey model. Multivariable grey model MGM (1, n) was improved to increase accuracy and integrity (Cui and Liu, 2008; He and Luo, 2009; Xiong and Dang, 2011). Grey model GM (1, 1) was extended to multivariable model, and the single variable grey model GM (1, 1) with integer order was extended to grey model FGM (1, 1) with fractional order accumulation. The work firstly analyzed fractional order AGO and IAGO of single variable. Then, background value formula of multivariable model FMGM (1, n) was deduced based on discrete exponential function optimization. After that, the work established multivariable grey model FMGM (1, n) with fractional order accumulation, deriving model parameter estimation based on least square method. By taking fractional order and minimum average relative error as design variable and object function, the work established optimal model and wrote solution program in Matlab. As natural promotion of single variable model FGM (1, 1), multivariable grey model FMGM (1, n) with fractional order accumulation was to reflect interaction of variables. At last, the numerical examples were given to indicate correctness and effectiveness of the model.
2. FRACTIONAL ORDER MULTIVARIABLE GREY MODEL BASED ON BACKGROUND VALUE GENERATED

Definition 1: If original sequence $x^{(0)} = [x^{(0)}(1), \cdots, x^{(0)}(j), \cdots, x^{(0)}(m)]$, then r-order accumulation generation will be expressed as follows.

$$x^{(r)} = [x^{(r)}(1), \cdots, x^{(r)}(j), \cdots, x^{(r)}(m)]$$

where $m$ is sequence number of each variable.

$$x^{(r)}(k) = \sum_{i=1}^{k} x^{(r-i)}(j) = x^{(r)}(k-1) + x^{(r)}(k)$$

(1)

Based on matrix operation principle, it can be concluded that

$$x^{(r)} = A_r x^{(r-1)} = A_r A_r x^{(r-2)} = \cdots = A_r^n x^{(0)}$$

where first-order accumulation generation matrix

$$A_r = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

and

$$(a_{i,j_k})_{m,n} = \begin{cases} \frac{(k_i - j_k + r - 1)!}{(r-1)!} & i \geq j_k \\ 0 & i < j_k \end{cases}$$

(2)

In Equation (2), $r$ is extended from integer to fraction, thus deriving fractional order AGO matrix $A_r^*$ satisfies $(A_r^*)^{-1} = A_r^{r-2}, A_r^{r-2} = \Gamma (r-1)!$, and $x^{(0)} = A_r^* A_r^{r-1} = A_r^{r-1} x^{(0)}$. $A_r^{r-1}$ is defined as r-order IAGO matrix. $x^{(r)}$ can be restored to original sequence $x^{(0)}$ using $A_r^{r-1}$.

In Equation (1), the coefficient of $x^{(0)}(j)$ is expressed as follows.

$$a_r = \frac{(k-j+1)(k-j+2) \cdots (k-j+r-1)}{(r-1)!} = \frac{\Gamma(r+k-1)}{\Gamma(k-j+1) \Gamma(r)}$$

where $k = 1, 2, \cdots, m$.

Therefore,

$$x^{(r)} = \sum_{j=1}^{m} \frac{\Gamma(r+k-1)}{\Gamma(k-j+1) \Gamma(r)} x^{(0)}(j)$$

(3)

where, $k = 1, 2, \cdots, m$, $\Gamma$ is Gamma function.

When $r$ is integer, Equation (1) will be suitable. Besides, Equation (3) can be used to calculate fractional order $x^{(r)}(k)$.

Let $x^{(0)} = [x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(m)]$, where $n$ is variable number, $x^{(r)}$ r-order GAO sequence of $x^{(0)}$.

$$x^{(r)}(k) = \sum_{j=1}^{m} \frac{\Gamma(r+k-1)}{\Gamma(k-j+1) \Gamma(r)} x^{(0)}(j)$$

(4)

FMGRM (1, n) model is n-element and r-order differential equation set.

$$\begin{bmatrix} \frac{dx^{(0)}}{dt} \\ \frac{dx^{(0)}}{dt} \\ \vdots \\ \frac{dx^{(0)}}{dt} \end{bmatrix} = a_1 x^{(0)} + a_2 x^{(0)} + \cdots + a_n x^{(0)} + b_1$$

(5)

It is denoted that $X^{(0)} = (x^{(0)}(k), x^{(0)}(k), \cdots, x^{(0)}(k))^T$; $X^{(r)}(k) = (x^{(r)}(k), x^{(r)}(k), \cdots, x^{(r)}(k))^T$; $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.
Equation (5) can be expressed as

$$\frac{dX_{r}^{(i)}}{dt} = AX_{r}^{(i)} + B$$  (6)

The first component \( x_{r}^{(i)}(1) \) of Sequence \( x_{r}^{(i)}(j) \) is taken as the initial condition of grey differential equation (\( x_{r}^{(i)}(1) = x_{0}^{(i)}(1) \)). Continuous time response of Equation (6) is expressed as follows.

$$X_{r}^{(i)}(t) = e^{\alpha t}X_{r}^{(i)}(m) + A \left( e^{\alpha t} - I \right) B$$  (7)

where \( e^{\alpha t} = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} t^k \), \( I \) is unit matrix.

To distinguish \( A \) and \( B \), Equation (5) is dispersed to derive Equation (8).

$$x_{r}^{(i-1)}(k) = \sum_{j=1}^{n} a_{j} x_{r}^{(i)}(k-j) + b_{i}$$  (8)

where

$$Z = \begin{bmatrix}
\frac{1}{2} (x_{r}^{(i)}(2) + x_{r}^{(i)}(0)) & \frac{1}{2} (x_{r}^{(i)}(2) + x_{r}^{(i)}(0)) & \cdots & \frac{1}{2} (x_{r}^{(i)}(2) + x_{r}^{(i)}(0)) \\
\frac{1}{2} (x_{r}^{(i)}(3) + x_{r}^{(i)}(1)) & \frac{1}{2} (x_{r}^{(i)}(3) + x_{r}^{(i)}(1)) & \cdots & \frac{1}{2} (x_{r}^{(i)}(3) + x_{r}^{(i)}(1)) \\
\frac{1}{2} (x_{r}^{(i)}(m) + x_{r}^{(i)}(m-1)) & \frac{1}{2} (x_{r}^{(i)}(m) + x_{r}^{(i)}(m-1)) & \cdots & \frac{1}{2} (x_{r}^{(i)}(m) + x_{r}^{(i)}(m-1)) \\
\end{bmatrix}
$$

It is denoted that \( a_{i} = [a_{i1}, a_{i2}, \ldots, a_{in}]^{T} \) (\( i = 1, 2, \ldots, n \)). Least square method is used to derive \( \hat{\alpha}_{i} \),

$$\hat{\alpha}_{i} = (Z^{T} Z)^{-1} Z^{T} Y_{i}, i = 1, 2, \ldots, n$$

Then the discrimination values of \( A \) and \( B \) are expressed as Equation (10).

$$\hat{A} = \begin{bmatrix}
\hat{a}_{i1} & \hat{a}_{i2} & \cdots & \hat{a}_{in} \\
\hat{a}_{i1} & \hat{a}_{i2} & \cdots & \hat{a}_{in} \\
\cdots & \cdots & \cdots & \cdots \\
\hat{a}_{i1} & \hat{a}_{i2} & \cdots & \hat{a}_{in} \\
\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}
\hat{b}_{i1} \\
\hat{b}_{i2} \\
\cdots \\
\hat{b}_{in} \\
\end{bmatrix}$$  (10)

The calculation value of NFMRM (1, n) model is as follows.

$$\hat{X}_{r}^{(i)}(j) = e^{\hat{\alpha}_{i} t} \underbrace{X_{r}^{(i)}(0) + \hat{A} (e^{\hat{\alpha}_{i} t} - I) \hat{B}}_{\hat{B}}$$  (11)

In Equation (11), the first component of original data sequence is taken as initial condition of grey differential equation. Equation (11) is reduced to obtain the fitting values of original data. Using \( r \)-order IGAT matrix, \( \hat{X}_{r}^{(i)} \) is reduced to original sequence \( X_{r}^{(i)} \).

The absolute and relative errors of the \( i \)-th variable are defined as Equations (12) and (13).

$$e_{i}(k) = \frac{\hat{X}_{r}^{(i)}(k) - X_{r}^{(i)}(k)}{X_{r}^{(i)}(k)} \times 100$$  (13)

Relative error average of the \( i \)-th variable is expressed as Equation (14).

$$\frac{1}{m} \sum_{k=1}^{m} |e_{i}(k)|$$  (14)

Average error of the whole data is expressed as Equation (15).

$$f = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} |e_{i}(k)|$$  (15)

The above is multivariable grey model FMGM (1, n) with fractional order accumulation. If \( r = 1 \), then the model will be transformed into multivariable grey model MGM (1, n). The model can be solved by the given fractional order.

3. FRACTIONAL ORDER MULTIVARIABLE GREY MODEL BASED ON DISCRETE EXPONENTIAL FUNCTION OPTIMIZATION OF BACKGROUND VALUE

The essential of average background value generation is to approximately calculate the area surrounded by \( t \) axis and \( x_{r}^{(i)}(k) \) in \([k-1, k]\) using trapezoid formula. When sequence data smoothly changes within small interval, then the constructed background values will be appropriate. When data sequence sharply changes, the
background values will produce large model error. Therefore, it is important to discuss background value construction of FMGM (1, 1).

To distinguish $A$ and $B$, Equation (5) is integrated in the interval $[k-1,k]$ to derive Equation (16).

\[
x_i^{(r)}(k) = \sum_{t=1}^{k} a_t x_i^{(r)}(k) dt + b_i \quad (i = 1,2,\ldots,n; k = 2,3,\ldots,m)
\]

It is denoted that $z_i^{(r)}(k) = \sum_{t=1}^{k} x_i^{(r)}(k) dt$. Traditional background values are calculated by trapezoidal area $z_i^{(r)}(k)$. When sequence data smoothly changes within small interval, then the constructed background values will be appropriate. When data sequence sharply changes, the background values will produce large model error. Therefore, background value of $x_i^{(r)}(k)$ in the interval $[k-1,k]$ can be expressed as $z_i^{(r)}(k) = \int_{k-1}^{k} x_i^{(r)}(k) dt$. The estimated parameter matrices $\hat{A}$ and $\hat{B}$ are more suitable for whitening of Equation (5).

Non-homogeneous exponential function is used to fit AGO sequence with fractional order. Let $x_i^{(r)}(k) = G_i e^{-\alpha_i k} + C_i$, where $G_i$ and $C_i$ are unknown quantities. The IAGO sequence is calculated by

\[
x_i^{(r)}(k) = x_i^{(r)}(k) - x_i^{(r)}(k-1) = G_i (1-e^{-\alpha_i}) e^{-\alpha_i (k-1)} = G_i e^{-\alpha_i (k-1)}
\]

where, $g_i = G_i (1-e^{-\alpha_i}) = -G_i a_i$.

It is known that

\[
x_i^{(r)}(k) = e^{-\alpha_i (k-1)}
\]

Then,

\[
a_i = -\ln \left( \frac{x_i^{(r)}(k)}{x_i^{(r)}(k-1)} \right) (k = 2,3,\ldots,m)
\]

Equation (8) is substituted in Equation (7) to derive Equation (19).

\[
G_i = \frac{e^{\alpha_i (k-1)} - [x_i^{(r)}(k)/x_i^{(r)}(k-1)]^{\alpha_i}}{1 - x_i^{(r)}(k)/x_i^{(r)}(k-1)}
\]

Equation (20) is obtained according to initial value condition $x_i^{(r)}(1) = G_i e^{-\alpha_i k_1} + C_i = G_i + C_i$.

\[
C_i = \frac{(x_i^{(r)}(k-1)^\alpha_i - x_i^{(r)}(k-1))}{(x_i^{(r)}(k-1)^\alpha_i - x_i^{(r)}(k-1)) - (x_i^{(r)}(k-1) - x_i^{(r)}(k))}
\]

Equations (18) and (20) are substituted in background value expression $\int_{k-1}^{k} x_i^{(r)}(k) dt$ to derive Equation (21).

\[
z_i^{(r)}(k) = \int_{k-1}^{k} x_i^{(r)}(k) dt = \frac{\alpha_i e^{-\alpha_i (k-1)^\alpha_i}}{\alpha_i - e^{-\alpha_i (k-1)}}
\]

Equation (9) can be transformed into Equation (22).

\[
\hat{a}_i = [a_1, a_2, \ldots, a_n] = (Z^T Z)^{-1} Z^T Y, i = 1,2,\ldots,n
\]

where, $Z = \begin{bmatrix} z_i^{(r)}(2) & z_i^{(r)}(3) & \cdots & z_i^{(r)}(2) & 1 \\ z_i^{(r)}(3) & z_i^{(r)}(3) & \cdots & z_i^{(r)}(3) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_i^{(r)}(m) & z_i^{(r)}(m) & \cdots & z_i^{(r)}(m) & 1 \end{bmatrix}$, $Y = [x_i^{(r)}(2), x_i^{(r-1)}(3), \ldots, x_i^{(r)}(m)]^T$.

Based on discrete exponential function optimization of background values, Equation (22) is used to establish multivariable grey model FMGM (1, n) with fractional order instead of Equation (9). Using IAGO, $\hat{x}_i^{(r)}$ can be reduced to original sequence $\hat{x}_i^{(r)}$ for model examination.
The model will be solved if fractional order is given. The most reasonable solution is obtained by optimization. Taking fractional order \( r \) and minimum average relative error as design variable and object function, FMGM (1, 1) model is established to solve the following optimization problem.

\[
\min f(r) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{k=0}^{m} \left( x_i^*(k) - x_i^0(k) / x_i^0(k) \right)^2
\]

(23)

Function \( RAGO.m \) was compiled by accumulated generation operator with fractional order. Function \( I.RAGO.m \) by inverse accumulated generation operator. These two functions were called by the above modeling method to write model solving function \( ImprovedFMGM.m. \) After writing main program \( main_ImprovedFMGM.m \), the work applied MATLAB optimization design function \( fmincon \) or other intelligent methods (e.g., chaos particle swarm optimization algorithm based on gray entropy quantum (Luo snd Che, 2014)) to call object function \( ImprovedFMGM.m \). At last, the parameters and model examination were output after solution of fractional order accumulated generation operator \( r \). MGM (1, n) model was examined by common methods including residual, correlation degree and posterior difference test. ImprovedFMGM (1,n) model was also tested by above methods.

4. APPLICATION EXAMPLES

Example 1: Tribological performance analysis of TiN film coating
Load is set as 600N; relative sliding speeds are set as 0.314m/s, 0.417m/s, 0.628m/s, 0.942m/s and 1.046m/s. Table 1 shows test data of TiN film coating (Liu and Xu, 2003).

<table>
<thead>
<tr>
<th>No.</th>
<th>Sliding speed (m/s)</th>
<th>Friction coefficient, ( \mu )</th>
<th>Wear rate ( \omega ) (x10^{-4} mg/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.314</td>
<td>0.251</td>
<td>7.5</td>
</tr>
<tr>
<td>2</td>
<td>0.471</td>
<td>0.258</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>0.628</td>
<td>0.265</td>
<td>8.5</td>
</tr>
<tr>
<td>4</td>
<td>0.942</td>
<td>0.273</td>
<td>9.5</td>
</tr>
<tr>
<td>5</td>
<td>1.046</td>
<td>0.288</td>
<td>11</td>
</tr>
</tbody>
</table>

According to discrete exponential function optimization of background value, the work established multivariable grey model MGM (1, 3) (r=1) of sliding speed, friction coefficient and wear rate. Model parameters were described as follows:

\[
A = \begin{bmatrix} 0.1358 & 8.1030 & 0.2468 \\ 0.0065 & 0.0435 & -0.0010 \\ 1.2085 & 22.3823 & -0.733 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1519 \\ 0.2494 \\ 7.2621 \end{bmatrix}
\]

Model value and relative error of friction coefficient \( \mu \) are (0.251, 0.258, 0.2648, 0.27234, 0.28045) and (0, 0.0019253, -0.077197, -0.24216, -2.6215); relative error averages of friction coefficient and model 0.58856 % and 4.3098%, respectively. Model test result is good.

According to discrete exponential function optimization of background value, the work established multivariable grey model FMGM (1, 3) of sliding speed, friction coefficient and wear rate. Model parameters were described as follows.

\[
A = \begin{bmatrix} 0.1413 & 8.44025 & -0.2570 \\ 0.0031 & -0.0230 & 0.0019 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1604 \\ 0.2528 \end{bmatrix}, \quad r = 1.0396.
\]

Model value and relative error of friction coefficient \( \mu \) are (0.251, 0.25803, 0.2648, 0.27234, 0.28562) and (0, 0.010668, -0.13612, 0.43166, -0.82657); relative error averages of friction coefficient and model were 0.281 % and 4.2226%, respectively. Model test result is good.

If background value is generated by mean (\( r = 1 \)), then relative error average of model will be 6.3312%. If optimal fractional order \( r = 1.278 \), then relative error average of model will be 5.969%.

Example 2: Tribological performance analysis of CrN film coating
Load is set as 600N; relative sliding speeds are set as 0.314m/s, 0.417m/s, 0.628m/s, 0.942m/s and 1.046m/s. Table 2 shows test data of CrN film coating (Liu and Xu, 2003).

<table>
<thead>
<tr>
<th>No.</th>
<th>Sliding speed (m/s)</th>
<th>Friction coefficient, ( \mu )</th>
<th>Wear rate ( \omega ) (x10^{-4} mg/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.314</td>
<td>0.323</td>
<td>9.5</td>
</tr>
<tr>
<td>2</td>
<td>0.471</td>
<td>0.331</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>0.628</td>
<td>0.339</td>
<td>10.5</td>
</tr>
<tr>
<td>4</td>
<td>0.942</td>
<td>0.350</td>
<td>12.5</td>
</tr>
<tr>
<td>5</td>
<td>1.046</td>
<td>0.373</td>
<td>17</td>
</tr>
</tbody>
</table>


According to discrete exponential function optimization of background value, the work established multivariable grey model MGM (1, 3) (\(r=1\)) of sliding speed, friction coefficient and wear rate. Model parameters were described as follows.

\[
A = \begin{bmatrix}
-0.0511 & 2.7282 & -0.0708 \\
0.0104 & 0.0390 & -0.0011 \\
2.5479 & 15.7177 & -0.6103
\end{bmatrix}, \quad B = \begin{bmatrix}
0.1909 \\
0.3222 \\
9.7968
\end{bmatrix}
\]

Model value and relative error of friction coefficient \(\mu\) are (0.323, 0.331, 0.33869, 0.34768, 0.35762) and (0.00023004, -0.092584, -0.6628, -4.1228); relative error averages of friction coefficient and model 0.97568 % and 5.484%, respectively. Model test result is good.

According to discrete exponential function optimization of background value, the work established multivariable grey model FMGM (1, 3) of sliding speed, friction coefficient and wear rate. Model parameters were described as follows.

\[
A = \begin{bmatrix}
-0.0707 & 2.8933 & -0.0752 \\
0.0069 & 0.0043 & 0.0017 \\
2.4442 & 15.6374 & -0.5567
\end{bmatrix}, \quad B = \begin{bmatrix}
0.2074 \\
0.3319 \\
10.0235
\end{bmatrix}, \quad r = 1.0977
\]

Model value and relative error of friction coefficient \(\mu\) are (0.323, 0.33113, 0.33774, 0.35245, 0.3722) and (0.03946, -0.3712, 0.69878, -0.21437); relative error averages of friction coefficient and model were 0.26476 % and 5.2654%, respectively. Model test result is good.

If background value is generated by mean \(r=1\), then relative error average of model will be 4.721%. If optimal frictional order \(r=0.026889\), then relative error average of model will be 4.4992%.

The above examples indicate necessity, adaptability and effectiveness of the models.

3. CONCLUSIONS

Background value greatly affects accuracy of grey model. Background value structure is one of the most effective methods to improve modeling accuracy of grey model. The work firstly analyzed fractional order AGO and IAGO of single variable. Then, formula of multivariable AGO was deduced based on discrete exponential function optimization. After that, the work established multivariable grey model FMGM (1, n) with fractional order accumulation, deriving model parameter estimation based on least square method. By taking fractional order and minimum average relative error as design variable and object function, the work established optimal model and wrote solution program based on Matlab. As natural promotion of single variable model FGM (1, 1), multivariable state model FMGM (1, n) aimed at reflecting the interaction of variables. FMGM (1, n) is expansion and complement rather than replacement of FGM (1, 1) model. The numerical examples indicate adaptability and effectiveness of the models.

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